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NATURAL TRANSVERSE VIBRATION ANALYSIS OF EULER-BERNOULLI BEAMS RESTING ON FILONENKO-BORODICH ELASTIC FOUNDATIONS USING GENERALIZED INTEGRAL TRANSFORM METHOD

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Abstract

This study presents an analysis of natural transverse vibrations of an Euler-Bernoulli beam resting on a Filonenko-Borodich elastic foundation (EBBoFBEF) using Generalized Integral Transform Method (GITM). The study is important in design of foundation beams against resonance failures. Such failures occur when the excitation frequency coincides with the natural frequency. The governing equation for free harmonic vibrations of the EBBoFBEF is a homogeneous equation. GITM uses eigenfunctions for similar dynamic thin beam that correspond to the boundary conditions as mode shapes and integral kernels. Thus, no prior determination of the shape functions is needed. The orthogonality properties of the eigenfunctions simplify the resulting integration. The equation is reduced to an algebraic problem. Solutions for natural frequencies are obtained for three cases of cantilever, clamped-clamped, and simply supported ends. The solutions are obtained for the boundary conditions at each vibration mode. The natural frequencies in this study for the dimensionless foundation structure parameters ranging from 1 to 10,000 for the first parameter and representing the vertical modulus 0 to 2.5 for the second parameter representing the tension in the coupling membrane for the first five modes were identical with previous exact solutions. For clamped-clamped and cantilever EBBoFBEF, the frequency parameters obtained by GITM were identical to previous results. The study showed that elastic foundations increase the natural vibration frequencies. These findings offer valuable insights into the dynamic response and vibration characteristics of EBBoFBEF with implications for design safety against resonance

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1.0 INTRODUCTION

The response of a vibrating beam on elastic foundation (BoEF) represents an important problem in soil-structure interactions and structural dynamics. The determination of the natural vibration frequencies is important in their analysis against resonant failures, which take place at excitation frequencies coincident with natural frequencies. In order to model vibrating BoEF problem, it is fundamental to consider the structural behaviour of the beam, the foundation and the interaction between them. Euler-Bernoulli beam theory (EBBT) is the classical theory that has been extensively applied when the depth, h, to length, l, ratio does not exceed 0.05. The EBBT assumes the orthogonality hypothesis that the plane cross-section remains planeand orthogonal to the beam's mid plane before and after loading. This assumption scopes the EBBT to thin beams where transverse shear deformation is regarded. This work is focused on thin beams and uses EBBT.

Timoshenko beam theory (TBT) considers transverse shear and rotational inertia and is an improvement on EBBT. The TBT is applicable to moderately thick beams. However, the constant value of shear stress over the depth violates shear stress-free boundary condition and shear correction factors are introduced to correct the violation. The disadvantage of the TBT is that an accurate estimate of the shear correction factor is complex for different cross-sectional shapes.

Shear deformation theories have also been constructed by Ghugal [7], Sayyad and Ghugal [2] and others to ensure that shear stress-free boundary conditions are not violated and shear strains are considered. The simplest elastic foundation model is the Winkler model which simulates the foundation to be independent, vertical Hookean springs as depicted in Figure 1. The Winkler model assumes that foundation reaction $p(x)$ at arbitrary point on the beam is a linear proportion of the deflection $w(x)$ of the beam at that point only [3].

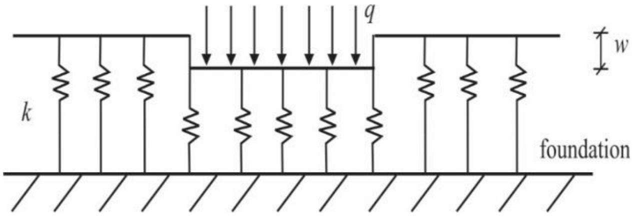

Figure 1: Winkler foundation

The Winkler foundation reaction model for beams is expressed simply by:

$$p(x) = kw(x) \tag{1}$$

where k is the single parameter used in the model, and it is called the Winkler parameter. It is equivalent to the spring constant, x is the longitudinal coordinate.

The Winkler model is not able to account for the continuous features of a foundation. This is because the Winkler parameters only accounts for the vertical stiffness, and presents a localized deflection due to an external load. Research efforts to improve the Winkler model have resulted in the introduction of coupling between the vertical springs. The kind of coupling determines the resulting model as a Filonenko-Borodich, Hetenyi or Pasternak model.

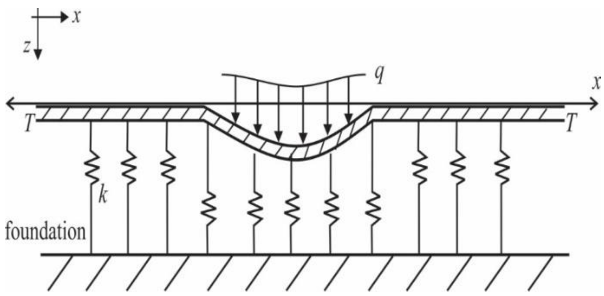

Figure 2: Filonenko-Borodich foundation

In the Filonenko-Borodich (FB) model shown in Figure 2, the coupling of the linear elastic springs is simulated using a stretched elastic membrane under a tensile force, T. The foundation reaction in the FB model is; for beam problems:

$$p(x) = kw(x) - T\frac{\partial^2}{\partial x^2}w(x) = kw(x) - Tw''(x) \tag{2}$$

where k and T are the two parameters of the model. BoEF equations are developed by incorporating the foundation interaction model in the beam theory. This study uses the EBBT and the two-parameter FB foundation model. The resulting equation is obtained by the incorporation of the FB foundation model in the vibrating EBBT equation.

Ogunbamike [4] used Galerkin method and Laplace transformations to obtain approximate solutions to the equation of motion of a uniform simply supported Rayleigh beam on FB foundation. Kanwal et al [5] explored Galerkin finite element methodology (GFEM) and variable separable methods for the eigenfrequencies determination of BoEF. Their research revealed that the shear layer flexural rigidity and foundation constants led to an increase in natural frequencies. Tazabekova et al [6] utilized variational

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iteration method (VIM) for accurate vibration frequencies of thin beams on Winkler foundations (TBoWF). The VIM equations were developed for various boundary conditions, and solved for clamped TBoWF. Al-Azzawi and Daud [7] investigated free vibration analysis of variable beam placed on variable Winkler foundations. Zhou [8] developed solutions for transversely vibrating thin beams on variable parameter Winkler foundation. Ike [9] utilized Fourier sine transformation methodology (FSM) for eigenfrequencies determination of prismatic Euler-Bernoulli BoWF foundation for Dirichlet boundary conditions. The study failed to consider EBBo2PEF, and non-simply supported ends. Ike [10] used Sumudu transform method for the eigenfrequencies analysis of thin beams, but did not consider thin BoEF. Ike [11] used Ritz variational methodology in the elastic stability solutions of slender beam on two-parameter elastic foundations (EBBo2PEF) under varieties of boundary conditions; but did not consider free vibration analysis. Ike [12] utilized the generalized integral transform method (GITM) to study the free vibration solutions of BoWF, for a variety of support conditions; but did not consider EBBo2PEF.

Rao and Raju [13] developed analytical solutions for vibratory analysis of EBBo2PEF with simply supported ends and clamped-clamped ends. Franciosi and Masi [14] derived accurate eigenfrequency solutions for free vibrations of EBBo2PEFs. Rahbar-Ranji and Shahbaztabar [15] used Legendre polynomials and Rayleigh Ritz method to derive exact solutions for free vibrations of EBBo2PEFs. Naidu and Rao [16] developed accurate eigenfrequency solutions for EBBo2PEFs. Cincin [17] and Cincin and Coskun [18] studied the "analysis of beam vibrations on partial elastic foundation using Adomian decomposition method" the "vibration analysis of a beam on a nonlinear elastic foundation" was investigated by [19]. Sahin [20] investigated "vibration of a composite elastic beam on an inhomogeneous elastic foundation." Zakeri and Attarnejad [21] studied "numerical free vibration analysis of higher-order shear deformable beams resting on two-parameter elastic foundation." Other significant studies on BoEF were presented by [22] and [23].

Recent studies on beams and BoEF are presented in Ike [24, 25, 26]. Ike [26] presented a variational derivation of the vibrating sinusoidal shear deformation beam problem and explored the exact solutions of the eigenfrequencies for the simply supported cases using finite sine transformation method. The work however did not consider elastic foundations effect. Ike [25] investigated the bending analysis of thick beam on two-parameter elastic foundation but did not consider vibrating cases. In a similar work, Ike [26] utilized Ritz method for deriving eigenfrequency solutions to thin beam on two-parameter foundation problems.

This work presents the GITM for the eigenfrequency analysis of EBBo2PEF of the FB type. GITM is used because of its demonstrated effectiveness as illustrated by [12], [27] and [28] in simplifying the solution of the governing partial differential equation (GPDE) by converting it to an integral equation. The objective of this study is to present GITM for the free vibration analysis of EBBo2PEF for various boundary conditions. The GITM is adopted because it does not need a prior determination of vibration shape functions as the shape functions are pre-selected as the vibrating eigenfunctions of identically supported beams.

2. METHODOLOGY

2.1 Governing Equation of Transverse Vibratory Motion (GETVM)

The GETVM is the fourth order partial differential equation (PDE)

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + k_w w - k_{EB}\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = q(x,t) \quad (3)$$

where $0 \le x \le l$, and $w(x, t)$ represents transverse deflection.

In Equation (3), l is the beam's span, t is the time, k_w and k_{EB} are the two parameters of Filonenko-Borodich (FB) foundation, ρ denotes mass density of the beam, A denotes beam's cross-sectional area, E denotes Young's modulus of the beam material, I denotes second moment of inertia of the beam cross-section, $q(x, t)$ is the excitation force which is transversely applied.

For prismatic beams, EI would not vary along the longitudinal coordinate, and for free vibrations, the excitation force $q(x, t)$ is absent, the PDE simplifies to a homogeneous partial differential equation:

$$EI\frac{\partial^4 w}{\partial x^4} + k_w w - k_{FB}\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \quad (4)$$

For sinusoidal response with phase, $w(x, t)$ can be expressed as:

$$w(x,t) = W(x)\sin(\omega_n t + \phi) \quad (5)$$

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$W(x)$ denotes modal displacement, ω_n denotes natural frequency, ϕ is the phase.

Then, substituting Equation (5) into Equation (4) and simplifying gives:

$$\left(EI\frac{d^4W}{dx^4}+k_wW(x)-k_{FB}\frac{d^2W(x)}{dx^2}-\rho A\omega_n^2W(x)\right)\sin(\omega_nt+\phi)=0 \quad (6)$$

Alternatively, dividing by EI and simplifying gives:

$$W^{iv}(x)-\frac{k_{FB}}{EI}W''(x)+\left(\frac{k_w}{EI}-\frac{\rho A\omega_n^2}{EI}\right)W(x)=0 \quad (7)$$

Let $\frac{k_w}{EI}=\bar{K}_w,\quad \frac{k_{FB}}{EI}=\bar{K}_{FB},\quad \frac{\rho A\omega_n^2}{EI}=\Omega_n^4$ (8)

$$W^{iv}(x)-\bar{K}_{FB}W''(x)+\left(\bar{K}_w-\Omega_n^4\right)W(x)=0 \quad (9)$$

2.2. Generalized Integral Transform of the GETVM

In the GITM, $W(x)$ is expressed as the infinite series summation of linear combinations of eigenfunctions $g_n(x)$ of transversely vibrating thin beams with equivalent boundaries [12]. Thus,

$$W(x)=\sum_{n=1}^{\infty}c_ng_n(x) \quad (10)$$

c_n denote generalized parameters of the deflection function.

This gives the generalized integral transformation of the ordinary differential equation (ODE) as:

$$\int_0^l\left(\frac{d^4}{dx^4}\sum_{n=1}^{\infty}c_ng_n(x)-\bar{K}_{FB}\frac{d^2}{dx^2}\sum_{n=1}^{\infty}c_ng_n(x)+\left(\bar{K}_w-\Omega_n^4\right)\sum_{n=1}^{\infty}c_ng_n(x)\right)g_m(x)dx=0 \quad (11)$$

$$\sum_{n=1}^{\infty}c_n\int_0^l\left(g_n^{iv}(x)g_m(x)-\bar{K}_{FB}\,g_n''(x)g_m(x)+\left(\bar{K}_w-\Omega_n^4\right)g_n(x)g_m(x)\right)dx=0 \quad (12)$$

From the additive properties of integration,

$$\sum_{n=1}^{\infty}c_n\left\{\int_0^l g_n^{iv}(x)g_m(x)dx-\bar{K}_{FB}\int_0^l g_n''(x)g_m(x)dx+\left(\bar{K}_w-\Omega_n^4\right)\int_0^l g_n(x)g_m(x)dx\right\}=0 \quad (13)$$

Let,

$$\int_0^l g_n^{iv}(x)g_m(x)dx=I_{nm_1};\quad \int_0^l g_n''(x)g_m(x)dx=I_{nm_2};\quad \int_0^l g_n(x)g_m(x)dx=I_{nm_3} \quad (14)$$

Then,

$$\sum_{n=1}^{\infty}c_n\left\{I_{nm_1}-\bar{K}_{FB}I_{nm_2}+\left(\bar{K}_w-\Omega_n^4\right)I_{mn_3}\right\}=0 \quad (15)$$

For eigenfunctions of vibrating beams,

$$g_n^{iv}(x)=\alpha_n^4g_n(x) \quad (16)$$

Then,

$$I_{nm_1}=\alpha_n^4\,I_{nm_3} \quad (17)$$

Hence,

$$\sum_{n=1}^{\infty}c_n\left\{\alpha_n^4I_{nm_3}-\bar{K}_{FB}I_{nm_2}+\left(\bar{K}_w-\Omega_n^4\right)I_{mn_3}\right\}=0 \quad (18)$$

For nontrivial solutions,

$$\alpha_n^4I_{nm_3}-\bar{K}_{FB}I_{nm_2}+\left(\bar{K}_w-\Omega_n^4\right)I_{mn_3}=0 \quad (19)$$

Solving for Ω_n^4,

$$\Omega_n^4=\alpha_n^4+\bar{K}_w-\bar{K}_{FB}\frac{I_{nm_2}}{I_{nm_3}}=\frac{\rho A\omega_n^2}{EI} \quad (20)$$

Then, ω_n is obtained as:

$$\omega_n^2=\frac{EI}{l^4\rho A}\left(\alpha_n^4+\bar{K}_w-\bar{K}_{FB}\frac{I_{nm_2}}{I_{nm_3}}\right)l^4 \quad (21)$$

$$\omega_n=\sqrt{\frac{EI}{\rho Al^4}}\left(\left(\alpha_n^4+\bar{K}_w-\bar{K}_{FB}\frac{I_{nm_2}}{I_{nm_3}}\right)l^4\right)^{1/2}=\frac{\lambda_n^2}{l^2}\sqrt{\frac{EI}{\rho A}} \quad (22)$$

$$\lambda_n^2=\sqrt{\left(\alpha_n^4l^4+\bar{K}_wl^4-\bar{K}_{FB}\frac{I_{nm_2}}{I_{nm_3}}l^4\right)} \quad (23)$$

λ_n denotes frequency parameter at the nth mode.

3. RESULTS AND DISCUSSION

3.1 Case 1: EBBo2PEF with simply supported ends ($x = 0$, $x = l$)

The boundary conditions (BCs) for a simply supported EBBo2PEF shown in Figure 3 are:

$$w(0,t) = w(l,t) = w''(0,t) = w''(l,t) = 0 \quad (24)$$

Hence,

$$W(0) = W''(0) = W(l) = W''(l) = 0 \quad (25)$$

Then eigenfunctions for a simply supported vibrating thin beam problem is $g_n(x) = \sin\alpha_n x$

$$W(x) = \sum_{n=1}^{\infty} c_n \sin\alpha_n x \quad (26)$$

where α_n are the roots of:

$$\sin\alpha_n l = 0 \quad (27)$$

Thus, $\alpha_n l = n\pi \qquad n = 1, 2, 3, 4, \ldots$ (28)

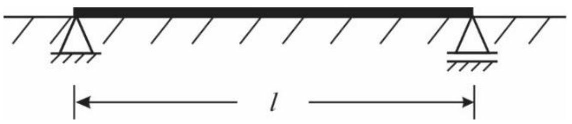

Figure 3: Simply supported Euler-Bernoulli beam on two-parameter elastic foundation

Using Equation (16),

$$I_{nm_1} = \alpha_n^4\, I_{nm_3} \quad (29)$$

$$I_{nm_3} = \int_0^l \sin\alpha_n x \sin\alpha_m x\, dx = 0 \quad \text{if } n \neq m \quad (30)$$

$$I_{nm_3} = \int_0^l \sin^2\alpha_n x\, dx = \int_0^l \sin^2\left(\frac{n\pi x}{l}\right) dx; \quad \text{if } n = m \quad (31)$$

$$g_n''(x) = -\alpha_n^2 g_n(x) \quad (32)$$

$$I_{nm_2} = \int_0^l g_n''(x) g_m(x) dx = -\alpha_n^2 \int_0^l g_n g_n dx \quad (33)$$

$$I_{nm_2} = -\alpha_n^2 \int_0^l g_n^2 dx = -\alpha_n^2 \int_0^l \sin^2\frac{n\pi x}{l} dx \quad (34)$$

Then,

$$\frac{\rho A\omega_n^2}{EI} = \alpha_n^4 + \bar{K}_w - \bar{K}_{FB}\left(-\alpha_n^2\right) = \alpha_n^4 + \bar{K}_w + \alpha_n^2\bar{K}_{FB} =$$

$$\alpha_n^4 + \bar{K}_w + \left(\frac{n\pi}{l}\right)^2 \bar{K}_{FB} \quad (35)$$

$$\omega_n^2 = \frac{EI}{\rho A}\left(\alpha_n^4 + \bar{K}_w + \frac{(n\pi)^2 K_{FB}}{l^2}\right) =$$

$$\frac{EI}{\rho A l^4}\left(\alpha_n^4 l^4 + \bar{K}_w l^4 + (n\pi)^2 \bar{K}_{FB} l^2\right) \quad (36)$$

$$\omega_n = \sqrt{\frac{EI}{\rho A l^4}}\sqrt{\left((n\pi)^4 + \bar{K}_w l^4 + (n\pi)^2 \bar{K}_{FB} l^2\right)} =$$

$$\frac{\lambda_n^2}{l^2}\sqrt{\frac{EI}{\rho A}} \quad (37)$$

$$\lambda_n^2 = \left((n\pi)^4 + \bar{K}_w l^4 + (n\pi)^2 \bar{K}_{FB} l^2\right)^{1/2} \quad (38)$$

Let $\hat{K}_w = \bar{K}_w l^4; \quad \hat{K}_{FB} = \dfrac{\bar{K}_{FB} l^2}{\pi^2}$ (39)

Then,

$$\lambda_n = \left((n\pi)^4 + \hat{K}_w + n^2\pi^4 \hat{K}_{FB}\right)^{1/4} \quad (40)$$

The non-dimensional transverse vibration parameters λ_n are determined for the first five modes n = 1, 2, 3, 4, 5 and for $\hat{K}_w = 1, 10, 100, 1000, 10000$ and $\hat{K}_{FB} = 0, 0.5, 1.0$ and 2.5 and presented in Table 1.

Table 1 also presents the previous values of λ_n determined for an identical problem [15]. Table 1 shows that the present GITM results are identical with previous results by [15]. Equation (40) was used to compute the transverse vibration frequencies for the first five vibration modes for EBBoWF by considering that for Winkler foundations, $\hat{K}_{FB} = 0$ in the equation. Table 2 illustrates that the present results can be used to calculate the eigenfrequencies of transverse vibration of EBBoWF. Table 2 further demonstrates that the present GITM results agree closely with previous results by [8].

3.2 Case 2: EBBo2PEF with clamped ends

The BCs for the EBBo2PEF having both ends (x =0, x = l) clamped as illustrated in Figure 4 are:

$$W(0) = W(l) = W'(0) = W'(l) = 0 \quad (41)$$

The eigenfunction is:

$$g_n(x) = (\cos\alpha_n x - \cosh\alpha_n x) - \beta_n(\sin\alpha_n x - \sinh\alpha_n x) \quad (42)$$

where $\beta_n = \dfrac{\cos\alpha_n l - \cosh\alpha_n l}{\sin\alpha_n l - \sinh\alpha_n l}$ (43)

and α_n is the nth root of the transcendental equation

$$\cos\alpha_n l \cosh\alpha_n l = 1 \quad (44)$$

The roots of Equation (59) are, from Mathematica software,

$$\alpha_1 l = 4.73004, \quad \alpha_2 l = 7.85321, \quad \alpha_3 l = 10.9956,$$
$$\alpha_4 l = 14.13717, \quad \alpha_n l = \pi(n + 0.5) \quad (45)$$

$$g_n''(x) = -\alpha_n^2 [(\cos\alpha_n x + \cosh\alpha_n x) - \beta_n(\sin\alpha_n x + \sinh\alpha_n x)] \quad (46)$$

Using Equation (40), λ_n are calculated for the first five vibration modes and for $\hat{K}_w = 1, 10, 100, 1000, 10000$ and $\hat{K}_{FB} = 0,\ 0.5,\ 1.0,\ 2.5$ and presented in Table 3. Table 3 demonstrates that the present GITM results for clamped-clamped EBBo2PEF are identical with previous solutions presented by [14] and [15].

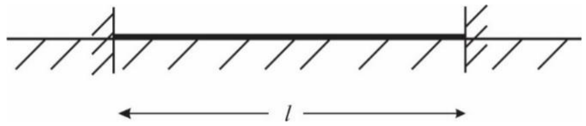

Figure 4: EBBo2PEF with clamped-clamped ends

3.3 Case 3: EBBo2PEF fixed supported at *x* = 0 and free at *x* = *l*

The eigenfunction for a cantilever EBBo2PEF shown in Figure 5 is:

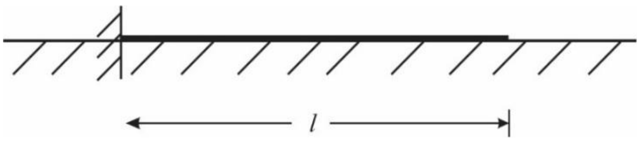

Figure 5: Cantilever EBBo2PEF

$$g_n(x) = (\cosh\alpha_n x - \cos\alpha_n x) - \beta_n(\sinh\alpha_n x - \sin\alpha_n x) \quad (47)$$

$$\beta_n = \frac{\cosh\alpha_n l + \cos\alpha_n l}{\sinh\alpha_n l + \sin\alpha_n l} \quad (48)$$

where α_n are the roots of:

$$\cosh\alpha_n l \cos\alpha_n l = -1 \quad (49)$$

The eigenvalues of the transcendental equation are obtained with the aid of computer software tools such as Mathcad, Wolfram Mathematica and iterative techniques.

$$\alpha_1 l = 1.87510,\ \alpha_2 l = 4.69409,\ \alpha_3 l = 7.85476,$$
$$\alpha_4 l = 10.99554,\ \alpha_n l = \left(\frac{2n-1}{2}\right)\pi \quad \text{for} \quad n \geq 5 \quad (50)$$

For dimensionless transverse vibration frequency parameters for a cantilever EBBo2PEF are calculated for $\hat{K}_w = 0,\ 1,\ 100,\ 10,000$ and $\hat{K}_{FB} = 0,\ 0.5,\ 1.0,\ 2.5$ and presented in Table 4. Table 4 shows that the present GITM results are in close agreement with previous results by [13] and [16].

Table 1: Dimensionless transverse vibration frequency parameters λ_n for first five modes of simply supported EBBo2PEF, $\hat{K}_w = \bar{K}_w l^4$

Foundation parameters		λ_1		λ_2		λ_3		λ_4		λ_5	
$\hat{K}_w$	$\hat{K}_{FB}$	Present	[15]	Present	[15]	Present	[15]	Present	[15]	Present	[15]
1	0	3.1496	3.1496	6.28426	6.28426	9.4251	9.4251	12.5665	12.5665	15.7080	15.7080
	0.5	3.4827	3.4827	6.4718	6.4718	9.5533	9.5533	12.6635	12.6635	15.7860	15.7860
	1.0	3.7408	3.7408	6.6445	6.6445	9.6766	9.6766	12.7584	12.7584	15.8628	15.8628
	2.5	4.3002	4.3002	7.0947	7.0947	10.0206	10.0206	13.0310	13.0310	16.0868	16.0868
10	0	3.2193	3.2193	6.2932	6.2932	9.4277	9.4277	12.5676	12.5676	15.7086	15.7086
	0.5	3.5347	3.5347	6.4801	6.4801	9.5560	9.5560	12.6646	12.6646	15.7865	15.7865
	1.0	3.7830	3.7830	6.6522	6.6522	9.6791	9.6791	12.7595	12.7595	15.8634	15.8634
	2.5	4.3282	4.3282	7.1010	7.1010	10.0229	10.0229	13.0320	13.0320	16.0873	16.0873
100	0	3.7484	3.7484	6.3816	6.3816	9.4545	9.4545	12.5790	12.5790	15.7144	15.7144
	0.5	3.9608	3.9608	6.5613	6.5613	9.5816	9.5816	12.6757	12.6757	15.7923	15.7923
	1.0	4.1437	4.1437	6.7273	6.7273	9.7038	9.7038	12.7703	12.7703	15.8690	15.8690
	2.5	4.5824	4.5824	7.1630	7.1630	10.0451	10.0451	13.0421	13.0421	16.0927	16.0927
1000	0	5.7556	5.7556	7.1121	7.1121	9.7102	9.7102	12.6905	12.6905	15.7721	15.7721
	0.5	5.8184	5.8184	7.2438	7.2438	9.8277	9.8277	12.7847	12.7847	15.8491	15.8491
	1.0	5.8793	5.8793	7.3686	7.3686	9.9412	9.9412	12.8770	12.8770	15.9250	15.9250
	2.5	6.0513	6.0513	7.7095	7.7095	10.2601	10.2601	13.1424	13.1424	16.1464	16.1464
10,000	0	10.0243	10.0243	10.5687	10.5687	11.5652	11.5652	13.6716	13.6716	16.3167	16.3167
	0.5	10.0363	10.0363	10.4122	10.4122	11.6354	11.6354	13.7472	13.7472	16.3863	16.3863
	1.0	10.0483	10.0483	10.4550	10.4550	11.7043	11.7043	13.8216	13.8216	16.4551	16.4551
	2.5	10.0842	10.0842	10.5806	10.5806	11.9042	11.9042	14.0378	14.0378	16.6563	16.6563

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Table 2: Dimensionless transverse vibration frequency parameters λ_n for first five modes for simply supported EBBoWEF

Foundation parameters		λ_1		λ_2		λ_3		λ_4		λ_5	
$\hat{K}_w$	$\hat{K}_{FB}$	Present	[8]	Present	[8]	Present	[8]	Present	[8]	Present	[8]
10	0	3.2193	3.220	6.2932	6.293	9.4277	9.427	12.5676	12.568	15.7086	15.709
100	0	3.7484	3.748	6.3816	6.382	9.4545	9.454	12.5790	12.579	15.7144	15.715
1000	0	5.7556	5.755	7.1121	7.112	9.7102	9.710	12.6905	12.690	15.7721	15.773

Table 3: Dimensionless transverse vibration frequency parameters λ_n for first five modes of EBBo2PEF with clamped-clamped ends

Foundation parameters		λ_1			λ_2			λ_3			λ_4		λ_5	
$\hat{K}_w$	$\hat{K}_{FB}$	Present	[15]	[14]	Present	[15]	[14]	Present	[15]	[14]	Present	[15]	Present	[15]
1	0	4.7324	4.7324	4.73	7.8537	7.8537	7.85	10.9958	10.9958	11	14.1372	14.1372	17.2788	17.2788
	0.5	4.8691	4.8691	4.87	7.9683	7.9683	7.97	11.0864	11.0864	11.09	14.2116	14.2116	17.3416	17.3416
	1.0	4.9946	4.9946	4.99	8.0780	8.0780	8.08	11.1748	11.1748	11.17	14.2847	14.2847	17.4037	17.4037
	2.5	5.3200	5.3200	5.32	8.3815	8.3815	8.38	11.4280	11.4280	11.43	14.4977	14.4977	17.5861	17.5861
10	0	4.7535	4.7535		7.8584	7.8584		10.9975	10.9975		14.1380	14.1380	17.2792	17.2792
	0.5	4.8885	4.8885		7.9728	7.9728		11.0880	11.0880		14.2124	14.2124	17.3420	17.3420
	1.0	5.0125	5.0125		8.0822	8.0822		11.1765	11.1765		14.2855	14.2855	17.4041	17.4041
	2.5	5.3350	5.3350		8.3853	8.3853		11.4295	11.4295		14.4984	14.4984	17.5865	17.5865
100	0	4.9504	4.9504	4.95	7.9043	7.9043	7.90	11.0143	11.0143	11.01	14.1460	14.1460	17.2836	17.2836
	0.5	5.0707	5.0707	5.23	8.0168	8.0168	8.16	11.1045	11.1045	11.24	14.2202	14.2202	17.3463	17.3463
	1.0	5.1823	5.1823	5.54	8.1245	8.1245	8.39	11.1925	11.1925	11.43	14.2932	14.2932	17.4084	17.4084
	2.5	5.4773	5.4773	5.48	8.4232	8.4232	8.42	11.4446	11.4446	11.44	14.5058	14.5058	17.5907	17.5907
1000	0	6.2239	6.2239		8.3251	8.3251		11.1790	11.1790		14.2248	14.2248	17.3270	17.3270
	0.5	6.2857	6.2857		8.4218	8.4218		11.2653	11.2653		14.2978	14.2978	17.3893	17.3893
	1.0	6.3455	6.3455		8.5150	8.5150		11.3497	11.3497		14.3696	14.3696	17.4509	17.4509
	2.5	6.5136	6.5136		8.7768	8.7768		11.5918	11.5918		14.5790	14.5790	17.6319	17.6319
10,000	0	10.1228	10.1228	10.12	10.8392	10.8392	10.84	12.5260	12.5260	12.53	14.9493	14.9493	17.7442	17.7442
	0.5	10.1374	10.1374	10.16	10.8835	10.8835	10.94	12.5876	12.5876	12.68	15.0122	15.0122	17.8022	17.8022
	1.0	10.1518	10.1518	10.21	10.9272	10.9272	11.04	12.6483	12.6483	12.81	15.0744	15.0744	17.8597	17.8597
	2.5	10.1943	10.1943	10.41	11.0546	11.0546	11.38	12.8252	12.8252	13.21	15.2564	15.2564	18.0287	18.0287

Table 4: Dimensionless transverse vibration frequency parameters λ_1 for the first mode of vibration of cantilever EBBo2PEF

Foundation parameters		λ_1		
$\hat{K}_w$	$\hat{K}_{FB}$	Present	[13]	[16]
0	0	1.9141	1.9141	1.876
0	0.5	2.5191	2.5191	2.496
0	1.0	2.8623	2.8623	2.832
0	2.5	3.4859	3.4859	3.421
1	0	1.9488	1.9488	1.902
1	0.5	2.5346	2.5346	2.513
1	1.0	2.8729	2.8729	2.869
1	2.5	3.4918	3.4918	3.473
100	0	3.2634	3.2634	3.206
100	0.5	3.4415	3.4415	3.432
100	1.0	3.5955	3.5955	3.487
100	2.5	4.0448	4.0448	4.002
10,000	0	10.003	10.003	10.001
10,000	0.5	10.010	10.010	10.011
10,000	1.0	10.017	10.017	10.017
10,000	2.5	10.039	10.039	10.041

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4. CONCLUSION

This paper explored natural transverse vibration analysis of EBBo2PEF of the FB type using GITM. In general, the GDEE of vibrating EBBo2PEF is a non-homogeneous PDE for forced vibrations which simplifies to homogeneous PDE for natural vibrations. For harmonic vibrations, the GITM reduces the problem to algebra. In conclusion,

(i) GITM utilizes eigenfunctions of free vibrations of EB beams with identical end restraints to be integral kernels in formulating GDEE.

(ii) the eigenfunctions of the GITM are orthogonal functions which result in a simplification of the resulting calculus problems.

(iii) the GITM yields analytical solutions to the eigenvalue problem since the GDEE is satisfied everywhere on the domain and BCs are also satisfied by the shape functions.

(iv) the GITM has no need for a pre-determination of shape functions since shape functions are apriori determined as the eigenfunctions of a dynamic thin beam having identical equivalent end restraints.

(v) for simply supported EBBO2PEF, frequency parameters λ_n are identical with Rahbar-Ranji and Shahbaztabar [15] results. The GITM results for simply supported EBBo2PEF obtained by setting the foundation coupling parameters equal to zero coincides with the exact Zhou [8] solutions for eigenfrequencies of thin BoWF.

(vi) for clamped-clamped EBBo2PEF, the dimensionless transverse vibration frequency parameters were identical with results by [14] and [15].

(vii) for cantilever EBBo2PEF, λ_n were closely in agreement with previous results by [13] and [16].

(viii) the study showed that an elastic foundation increases natural frequencies of BoEF. The two-parameter FB foundation presents higher frequencies than the Winkler foundation for all vibratory modes. The cantilever EBBo2PEF is most affected by the two-parameter FB foundation.

REFERENCES

[1] Ghugal Y.M. "A single variable parabolic shear deformation theory for flexure and flexural vibration of thick isotropic beams." *3rd International Conference on Structural Engineering Mechanic and Computation, Cape Town, South Africa*, p. 77 – 78, 2007

[2] Sayyad A.S. and Ghugal Y.M. "Single variable refined beam theories for the bending, buckling and free vibration of homogeneous beams." *Applied and Computational Mechanics*, 10(2), p. 123 – 138, 2016. https://www.kme.zcu.cz/acm/acm/article/view/304/377

[3] Younesian D., Hosseinkhani A., Askari H. and Esmanizadeh E. "Elastic and viscoelastic foundations: a review on linear and nonlinear vibration modeling and applications." *Nonlinear Dynamics*, 97, p. 853 – 895, 2019. https://doi.org/10.1007/s11071-019-04977-9

[4] Ogunbamike O.K. "Dynamics of beams on Filonenko-Borodich foundations." *Coast Journal of the School of Science*, 2(2), p. 490 – 498, 2020. www.journals.oaustech.edu.ng

[5] Kanwal G., Ahmed N. and Nawaz R. "A comparative analysis of the vibrational behaviour of various beam models with different foundation designs." *Heliyon*, 10(5), p. e26491, 2024. https://doi.org/10.1011/j.heliyon.2024.e26491

[6] Tazabekova A., Adair D., Ibrayev A. and Kim J. "Free vibration calculations of an Euler-Bernoulli beam on an elastic foundation using He's variational iteration method." Inc Wu, W., Yu, H.S. (Eds) *Proceedings of China-Europe Conference on Geotechnical Engineering*, Springer Series in Geomechanics and Geoengineering. Springer Cham. pp 1022 – 1026, 2018. https://doi.org/10.1007/978-3-319-97115-5-30

[7] Al-Azzawi A.A. and Daud K.A. "Free vibration of non-prismatic beam on variable Winkler elastic foundation." *IOP Conference Series Materials Science and Engineering*, 737, p. 012025, 2020. https://doi.org/10.1088/1751-899x/737/01/012025

[8] Zhou D. "A general solution to vibrations of beams on variable Winkler foundation." *Computers and Structures*, 47(1), p. 83 – 90, 1993. https://doi.org/10.1011/0045-7949(93)90281-H

[9] Ike C.C. "Fourier sine transform method for the free vibration of Euler-Bernoulli beam resting on Winkler foundation." *International Journal of Darshan Institute Engineering Research and Emerging Technologies*, 7(1), p. 1 – 6, 2018. https://doi.org/10.32692/IJDI-ERET/7.1.2018.1801

[10] Ike C.C. "Sumudu transform method for finding the transverse natural harmonic vibration frequencies of Euler-Bernoulli beams." *ARPN Journal of Engineering and Applied Science*, 16(9), p. 903 – 911, 2021. https://www.arpnjournals.com

[11] Ike C.C. "Ritz variational method for buckling analysis of Euler-Bernoulli beams resting on two-parameter foundations." *Iraqi Journal of Civil Engineering*, 18(1), p. 26 – 49, 2024. https://doi.org/10.37650/ijce.2024/80103

[12] Ike C.C. "Free vibration of thin beams on Winkler foundations using generalized integral transform method." *Engineering and Technology Journal*, 41(11), p. 1286 – 1297, 2023. https://doi.org/10.37650/etj.2023.140343.1462

[13] Rao G.V. and Raju K.K. "Elegant and accurate closed form solutions to predict vibration and buckling behaviour of slender beams on Pasternak foundation." *Indian Journal of Engineering and Materials Sciences*, 9, p. 98 – 102, 2002. https://nopr.nister.res.in

[14] Franciosi C. and Masi A. "Free vibration of foundation beams on two-parameter elastic soil." *Computers and Structures*, 47(3), p. 419 – 426, 1993. https://doi.org/10.1016/0045-7949(93)90237-8

[15] Rahbar-Ranji A. and Shahbaztabar A. "Free vibration analysis of beams on Pasternak foundation using Legendre polynomials and Rayleigh-Ritz method." *Proceedings of Odessa Polytechnic University*, 3(53), p. 20 – 31, 2017. https://doi.org/10.15276/opu.3.53.2017.03

[16] Naidu N.R. and Rao G.V. "Vibrations of initially stressed uniform beams on two-parameter elastic foundation." *Computers and Structures*, 57(3), p. 941 – 943, 1995. https://doi.org/10.1016/0045-7949(95)00090-4

[17] Cincin C. "Analysis of vibration: Beam partially supported on elastic foundation using Adomian decomposition method. "MSc Thesis, Kocaeli University, Kocaeli, 2018.

[18] Cincin C. and Coskun S.B. "Analysis of beam vibration on partial elastic foundation using Adomian decomposition method." *Kocaeli Journal of Science and Engineering*, 2(2), p. 28 – 33, 2019. https://doi.org/10.34088/kojose.491345

[19] Karahan M.M.F. and Pakdemirli M. "Vibration analysis of a beam on a nonlinear elastic foundation." *Structural Engineering and Mechanics*, 62(2), p. 171 – 178, 2017. https://doi.org/10.12989/sem.2017.26.2.171

[20] Sahin O. "Vibration of a composite elastic beam on an inhomogeneous elastic foundation." *Journal of Applied Mathematics and computational Mechanics*, 19(2), p. 107 – 119, 2020. https://doi.org/10.17512/JAMCM.2020.209

[21] Zakeri M. and Attarnejad R. "Numerical free vibration analysis of higher-order shear deformable beams resting on two-parameter elastic foundation." *Journal of Computational Applied Mechanics*, 46(2), p. 117 – 131, 2015. https://doi.org/10.22059/jcamech.2015.55094

[22] Bahuguna A., Singh A.K. and Rawat K.S. "Forced vibration: Axially functionally graded thin beams on elastic foundation." *Webology*, 18(4), p. 2183 – 2190, 2021. https://www.webology.org

[23] Wieckowski Z. and Swiatkiewicz P. "Stress-based FEM in the problem of bending of Euler-Bernoulli and Timoshenko beams resting on elastic foundation." *Materials*, 14(2), p. 460, 2021. https://doi.org/10.3390/ma14020460

[24] Ike C.C. "Variational formulations of vibration equations of sinusoidal shear deformable beams and eigenfrequency solutions by finite sine transform method." *Nigerian Journal of Technology*, 43(3), p. 421 – 427, 2024. https://doi.org/10.4314/njt.v43i3.3

[25] Ike C.C. "Static flexure of a first order shear deformable thick beam resting on a two-parameter elastic foundation – Equilibrium formulation and closed term solutions." NIPES – *Journal of Science and Technology Research*, 7(1), p. 235 – 254, 2025. https://doi.org/10.37933/nipes/7.1.2025.19

[26] Ike C.C. "Ritz variational method for the free harmonic vibration solutions of slender beam on two-parameter elastic foundation." *Nigerian Journal of Technology*, 41(2), p. 193 – 201, 2025. https://doi.org/10.4314/njt.v44i2.3

[27] Ike C.C., Onyia M.E. and Rowland-Lato E.O. "Generalized integral transform method of bending and buckling analysis of rectangular thin plate with two opposite edges simply supported and other edges clamped." *Advances in Science, Technology and Engineering Systems Journal*, 6(1), p. 283 – 296, 2021. https://doi.org/10.25046/aj060133

[28] Ike C.C. "Generalized integral transform method for bending analysis of clamped rectangular thin plates." *Journal of Computational Applied Mechanics*, vol. 53, no. 4, pp 599 – 625, 2022. https://doi.org/10.22059/JCAMECH.2022.350620.768