Nigerian Journal of Technology (NIJOTECH)
Vol. 45, No. 1, March 2026, pp. 216- 227
www.nijotech.com

Print ISSN: 0331-8443
Electronic ISSN: 2467-8821
https://dx.doi.org/10.4314/njt.v45i1.16

MODELLING OF DYNAMICAL SYSTEMS: HYBRID BLOCK METHODS FOR STIFF INITIAL VALUE PROBLEMS

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ARTICLE HISTORY:
Received: October 11, 2024.
Revised: October 12, 2025.
Accepted: November 29, 2025.
Published: April 29, 2026.

KEYWORDS:
A-stability, Block methods, Dynamical systems, Hybrid methods, Interpolation and Collocation, Stiff IVPs.

ARTICLE INCLUDES:
Peer review

DATA AVAILABILITY:
On request from author(s)

EDITORS:
Chidozie Charles Nnaji

FUNDING:
None

Abstract

Stiffness is a common phenomenon in differential equations that describe models of dynamical systems. Stiff initial value problems (IVPs) in ordinary differential equations (ODEs) pose challenges in their computation due to rapidly varying timescales in their solution components. In this study, a family of high order hybrid block methods with enhanced stability for stiff IVPs is developed. The technique of interpolation and collocation is used to determine the parameters of the hybrid block methods. The family of methods is shown to attain A-stability of order $p \leq 14$. Numerical experiments conducted on stiff IVPs show that the proposed family of methods has superior accuracy when compared to existing methods in the literature.

HOW TO CITE:
Akai, U. P., Ante, J. E., Asuk, E. E., Essang, S. O., Francis, R. E. and Otobi, A. O. "Modelling of Dynamical Systems: Hybrid Block Methods for Stiff Initial Value Problems", *Nigerian Journal of Technology*, 2026. 45(1), pp. 216- 227. https://dx.doi.org/10.4314/njt.v45i1.16

1.0 INTRODUCTION

The analysis of most dynamical systems depends on solving differential equations that describe mathematical models representing these systems. However, certain classes of these models present significant challenges due to their distinctive mathematical properties, such as oscillation, singularities, and stiffness. Stiffness, in particular, frequently arises in applications like chemical reaction kinetics, electronic circuits, mechanical vibrations, and molecular dynamics [1], [2], [3], [4], [5]. Solving stiff initial value problems (IVPs) is computationally demanding because their solutions involve rapidly varying timescales. A numerical method for integrating stiff IVPs must be implicit, A-stable, and cannot exceed order p =2, (see [6]). Consequently, among linear multistep methods (LMMs), only the trapezoidal rule meets the A-stability criterion and has the smallest error constant (EC). Explicit LMMs are generally unsuitable for stiff IVPs, as they either fail or require impractically small step sizes, leading to inefficient computations. These constraints significantly narrow the selection of viable numerical methods for stiff IVPs, as conventional schemes struggle under the strict step-length restrictions imposed by stability requirements. Therefore, the development of numerical algorithms for stiff problems must carefully balance factors such as step-size selection, stability, accuracy, and computational efficiency. As a result, attention is given to constructing numerical schemes with optimal stability properties, minimal error constants, and large stability regions.

A linear system with constant coefficients is given as

$$y' = f(t, y(t)) = Ay + \phi(t), \quad f:\mathbf{R}\times\mathbf{R}^m \to \mathbf{R}^m, \quad y:\mathbf{R}\to\mathbf{R}^m, \quad y(t_0)=y_0 \qquad (1)$$

where A is an m-dimensional matrix with real coefficients and distinct eigenvalues, λ_j, $j = 1, 2, \cdots, m$, contained in the left half of the complex plane. The system (1) is stiff over the finite interval $a \le t \le b$ if for every t in the interval, the eigenvalues $\lambda_j, (j = 1, 2, \cdots, m)$ of the Jacobian matrix $J = \dfrac{\partial f}{\partial y}$ satisfy the following conditions:

i.) $\operatorname{Re}(\lambda_j) < 0$, $j = 1, 2, \cdots, m$ where $\operatorname{Re}(\lambda_j)$ is the real part of the complex root $\lambda_j, (j = 1, 2, \cdots, m)$;

ii.) the ratio $\varphi = \dfrac{\max|\lambda_j|}{\min|\lambda_j|} >> 1$, for $j = 1, 2, \cdots, m$ holds, and the parameter φ is referred to as the stiffness ratio, [2], [7].

The general solution to (1) is given as

$$y(t) = \sum_{j=1}^{m} k_j e^{\lambda_j t}\eta_j + \vartheta(t), \qquad (2)$$

where $k_j, (j = 1, 2, \cdots, m)$ are arbitrary constants, $\eta_j, (j = 1, 2, \cdots, m)$ are the eigenvectors corresponding to the eigenvalues λ_j, $(j = 1, 2, \cdots, m)$, and $\vartheta(t)$ is a particular integral. The general solution of (1) is made up of the transient and steady-state solutions. The transient part of the solution $\sum_{j=1}^{m} k_j e^{\lambda_j t}\eta_j \to 0$ faster as $t \to \infty$ for $\operatorname{Re}(\lambda_j) < 0$, so that the solution $y(t)$ approaches the steady state solution $\vartheta(t)$ asymptotically as $t \to \infty$. As $t \to \infty$, the transient part of (2) decays monotonically if $\lambda_j \in$ □ $j = 1, 2, \cdots, m$ and sinusoidal if $\lambda_j \in$ □ $j = 1, 2, \cdots, m$.

The development of multistep methods for tracking the solution of stiff IVPs has been an area of extensive research. Among these, Backward Differentiation Formulae (BDF) have been widely used due to their strong stability properties. BDF methods are A-stable for k = 1(1)2 and A(α)-stable for k = 3(1)6, making them effective for stiff problems. To further enhance their stability and accuracy, the BDF class was extended to the Extended Backward Differentiation Formulae (EBDF), which achieve A-stability for k = 1(1)3 and A(α)−stability for k = 4(1)8. Another important class of methods for stiff systems is the Enright formulae, which belong to the family of second derivative methods. These methods improve stability characteristics by incorporating second derivative terms, making them particularly useful for stiff problems requiring higher-order accuracy. Their design allows for better error control and improved efficiency, making them competitive alternatives to traditional BDF-based approaches. The Enright formula is given as

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k} b_j f_{n+j} + h^2 d_k f'_{n+k}. \qquad (3)$$

The design of the method was to attain zero-stability as well as stability at infinity. Thus, the method (3) is A-stable for k = 1(1)2 and A(α)−stable for k = 3(1)7 but unstable for k ≥ 8, [8]. One of the objectives of adding a second derivative term was to circumvent the Dahlquist order barrier on LMMs.

https://dx.doi.org/10.4314/njt.v45i1.16

To facilitate meaningful contribution to developing numerical methods with better accuracy, hybrid methods emerged as a promising alternative that incorporates function evaluations at off-node points. This new class of formulae was simultaneously proposed by by Gragg and Statter, [9]. Hybrid methods preserve some features of linear multistep methods while also incorporating a key trait of Runge-Kutta methods. This includes the ability to use data at points beyond the step points, [10], [11]. This flexibility allows the methods to achieve higher accuracy and improved stability in solving differential equations. As a result, hybrid methods are particularly useful for stiff and non-stiff problems where both efficiency and stability are crucial. A new class of hybrid methods that incorporates off-step point within a step was introduced in [5]. Hybrid methods with multi-derivative of the function and additional off-step points in the first derivative of the solution to improve the stability regions was proposed in [9]. Recently, new hybrid block methods formulated in variable step-size mode were proposed in [12]. The emergence of methods with hybrid points brought potential reduction of the number of steps in traditional LMM while achieving enhanced stability. This represents a significant advancement in the development of numerical algorithms for stiff IVPs. Nonetheless, a disadvantage with hybrid methods is that a predictor is often required in a predictor-corrector (P-C) approach. A setback in this approach is that it is not economical in terms of time since the number of iterations is not determined *a priori*. In the literature, further innovations have seen the P-C approach deployed in other fashions. Though this approach has enhanced stability, there are complexities involved in its usability. Hence new approaches have since emerged as alternative to the P-C technique.

Let the exact solution value $y(t_{n+i})$ of the differential equation (1) be denoted by the approximation y_{n+i} at the point $t_{n+i}=t_n+ih$. By specifying $Y_{m-i}=\left(y_{n-ik+1},y_{n-ik+2},\cdots,y_{n-ik+k}\right)$, $F(Y_{m-i})=\left(f_{n-ik+1},f_{n-ik+2},\cdots,f_{n-ik+k}\right)$ and $F'(Y_{m-i})=\left(f'_{n-ik+1},f'_{n-ik+2},\cdots,f'_{n-ik+k}\right)$, $i=0,1$, $(n=mk,\ m=0,1,\cdots)$, [13] considered a one-block k-point method as the matrix difference equation

$$A_0Y_m=A_1Y_{m-1}+hB_1F_{m-1}+hB_0F_m+h^2D_0F_m. \tag{4}$$

Interests in block methods have emerged as an alternative to compute solutions to stiff IVPs due to high computational cost and time associated with other techniques. In [14], [15], block methods for the integration of stiff initial value problems were developed. Block methods with hybrid schemes were developed in [16], [17], [18].

In this study, we aim to develop a family of hybrid block methods from (3) using interpolation and collocation techniques. The proposed methods are designed to achieve A-stability for orders up to $p \le 14$. By formulating the method in a continuous form and evaluating at specific points, a set of discrete schemes is derived, forming the hybrid block methods for solving Equation (1). The paper is structured as follows: Section 2 presents the construction of the hybrid block method, while Section 3 discusses its implementation and numerical results. Finally, Section 4 provides concluding remarks on the study.

2.0 CONSTRUCTION OF THE HYBRID BLOCK METHOD

Consider the numerical solution of (1) as a polynomial of the form

$$Y(t)=\sum_{j=0}^{q+s-1}\alpha_j\psi_j(t), \tag{5}$$

where α_j are the required coefficients to be determined and $\psi_j(t)$ the polynomial basis functions of degree $q+s+1$. Let the number of interpolation points q and the number of distinct collocation points s be respectively chosen to satisfy $q+s-1$. The proposed class of methods is constructed from (3) by specifying $\psi_j(t)=t^j$, $j=0,1,\cdots,k$, and by imposing the conditions

$$y_n=\alpha_0 \tag{6}$$

$$y'(t)=\sum_{j=1}^{q+s-1} j\alpha_j t^{j-1}=f_{n+v_i},\ v_i=\tfrac{i}{2},\ i=0,1,2,\cdots,2k \tag{7}$$

$$y''(t)=\sum_{j=2}^{q+s-1} j(j-1)\alpha_j t^{j-2}=f'_{n+v_i},\ \ v_i. \tag{8}$$

Interpolating at t_n and collocating at $v_i=\frac{i}{2}$, $i=0,1,2,\cdots,2k$ intra-step points leads to a system of $2k+3$ equations. The resultant system of $q+s-1$ equations from interpolation and collocation

https://dx.doi.org/10.4314/njt.v45i1.16

of the equations (6), (7) and (8) is given in compact form as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2\cdot(\frac{1}{2}) & 3\cdot(\frac{1}{2})^2 & 4\cdot(\frac{3}{2})^3 & \cdots & k\cdot(\frac{3}{2})^{k-1} & (k+1)\cdot(\frac{1}{2})^k & (k+2)\cdot(\frac{1}{2})^{k+1} & \cdots & (2k+2)\cdot(\frac{1}{2})^{2k+1} \\ 0 & 1 & 2 & 3 & 4 & \cdots & k & k+1 & k+2 & \cdots & 2k+2 \\ 0 & 1 & 2\cdot(\frac{3}{2}) & 3\cdot(\frac{3}{2})^2 & 4\cdot(\frac{3}{2})^3 & \cdots & k\cdot(\frac{3}{2})^{k-1} & (k+1)\cdot(\frac{3}{2})^k & (k+2)\cdot(\frac{3}{2})^{k+1} & \cdots & (2k+2)\cdot(\frac{3}{2})^{2k+1} \\ 0 & 1 & 2\cdot 2 & 3\cdot 2^2 & 4\cdot 2^3 & \cdots & k\cdot 2^{k-1} & (k+1)\cdot 2^k & (k+2)\cdot 2^{k+1} & \cdots & (2k+2)\cdot 2^{2k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2k & 3k^2 & 4k^3 & \cdots & k\cdot k^{k-1} & (k+1)\cdot k^k & (k+2)\cdot k^{k+1} & \cdots & (2k+2)\cdot k^{2k+1} \\ 0 & 0 & 2 & 3\cdot 2\cdot v_i & 4\cdot 3\cdot v_i^2 & \cdots & k\cdot(k-1)\cdot v_i^{k-2} & (k+1)\cdot k\cdot v_i^{k-1} & (k+1)\cdot(k+1)\cdot v_i^k & \cdots & (2k+2)\cdot k\cdot v_i^{2k+1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \vdots \\ \alpha_{2k+1} \\ \alpha_{2k+2} \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ \vdots \\ f_{n+k} \\ f'_{n+v_i} \end{pmatrix}$$

The system of equations generate the coefficients α_j that are replaced in (5). With some algebraic manipulations, the continuous formulation of the second derivative hybrid scheme is generated as

$$Y(t) = h\sum_{j=0}^{q+s-1} b_j(t) f_{n+v_i} + h^2 d_j(t) f'_{n+v_i}, \qquad (9)$$

where $v_i = \frac{i}{2}$, $i = 0,1,2,\cdots,k$ are intra-step points $t_n, t_{n+\frac{1}{2}}, t_{n+1}, \cdots, t_{n+\frac{2k-1}{2}}, t_{n+k}$.

The structure of the block method has matrices

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & b_{10} \\ 0 & 0 & \cdots & 0 & b_{20} \\ 0 & 0 & \cdots & 0 & b_{30} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_{k0} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1k} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2k} \\ b_{31} & b_{32} & b_{33} & \ddots & b_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & b_{k3} & \cdots & b_{kk} \end{pmatrix},$$

$$D_0 = \begin{pmatrix} d_{1k} & 0 & 0 & \cdots & 0 \\ 0 & d_{2k} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & d_{k-1k} & 0 \\ 0 & 0 & 0 & 0 & d_{kk} \end{pmatrix}.$$

Given that the matrices A_0 and A_1 are defined *a priori* from the structure of the hybrid block scheme (4), the matrix coefficients of the family of hybrid block methods are:

Two-Point Hybrid Block Method

$$B_1 = \begin{pmatrix} 0 & \frac{7}{48} \\ 0 & \frac{1}{6} \end{pmatrix}, \quad B_0 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{48} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix},$$

$$D_0 = \begin{pmatrix} -\frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix}$$

Four-Point Hybrid Block Method

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{367}{2880} \\ 0 & 0 & 0 & \frac{53}{360} \\ 0 & 0 & 0 & \frac{49}{320} \\ 0 & 0 & 0 & \frac{7}{45} \end{pmatrix}, \quad B_0 = \begin{pmatrix} \frac{421}{1440} & \frac{47}{480} & -\frac{29}{1440} & \frac{7}{2880} \\ \frac{4}{5} & \frac{2}{15} & -\frac{4}{45} & \frac{1}{120} \\ \frac{117}{160} & \frac{27}{160} & \frac{67}{160} & \frac{9}{320} \\ \frac{32}{45} & \frac{4}{15} & \frac{32}{45} & \frac{7}{45} \end{pmatrix},$$

https://dx.doi.org/10.4314/njt.v45i1.16

$$D_0 = \begin{pmatrix} -\frac{3}{32} & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & 0 \\ 0 & 0 & -\frac{3}{32} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Six-Point Hybrid Block Method

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{28549}{241920} \\ 0 & 0 & 0 & 0 & 0 & \frac{1027}{7560} \\ 0 & 0 & 0 & 0 & 0 & \frac{253}{1792} \\ 0 & 0 & 0 & 0 & 0 & \frac{136}{945} \\ 0 & 0 & 0 & 0 & 0 & \frac{7045}{48384} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{280} \end{pmatrix}, B_0 = \begin{pmatrix} \frac{112223}{483840} & \frac{5717}{26880} & -\frac{10621}{120960} & \frac{7703}{241920} & -\frac{403}{53760} & \frac{199}{241920} \\ \frac{97}{105} & \frac{1993}{7560} & -\frac{394}{945} & \frac{97}{840} & -\frac{23}{945} & \frac{19}{7560} \\ \frac{1485}{1792} & -\frac{2403}{8960} & \frac{17}{35} & \frac{3267}{8960} & -\frac{513}{8960} & \frac{47}{8960} \\ \frac{752}{945} & -\frac{2}{105} & \frac{1312}{945} & -\frac{158}{945} & -\frac{16}{105} & \frac{2}{189} \\ \frac{8375}{10752} & \frac{3125}{48384} & \frac{25625}{24192} & -\frac{625}{5376} & \frac{52205}{96768} & \frac{1375}{48384} \\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & \frac{41}{280} \end{pmatrix}, D_0 = \begin{pmatrix} -\frac{275}{2304} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{45}{128} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{275}{2304} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Eight-Point Hybrid Block Method

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{324901}{2903040} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58193}{453600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{23887}{179200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7703}{56700} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11395}{82944} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{31}{224} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{288533}{2073600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1978}{14175} \end{pmatrix}, B_0 = \begin{pmatrix} \frac{5912189}{36288000} & \frac{653203}{1814400} & -\frac{50689}{226800} & \frac{196277}{1451520} & -\frac{92473}{1451520} & \frac{95167}{4536000} & -\frac{7703}{1814400} & \frac{5741}{14515200} \\ \frac{14692}{14175} & \frac{660143}{1134000} & -\frac{15458}{14175} & \frac{22703}{45360} & -\frac{2984}{14175} & \frac{14773}{226800} & -\frac{898}{70875} & \frac{521}{453600} \\ \frac{1467}{1600} & -\frac{4707}{5600} & \frac{9059}{89600} & \frac{28143}{17920} & -\frac{11079}{22400} & \frac{3047}{22400} & -\frac{2223}{89600} & \frac{387}{179200} \\ \frac{12416}{14175} & -\frac{232}{567} & \frac{40064}{14175} & -\frac{908}{2835} & -\frac{19072}{14175} & \frac{3944}{14175} & -\frac{128}{2835} & \frac{209}{56700} \\ \frac{248375}{290304} & -\frac{19375}{72576} & \frac{143375}{72576} & -\frac{641875}{290304} & \frac{400565}{290304} & \frac{12875}{18144} & -\frac{3125}{36288} & \frac{3625}{580608} \\ \frac{738}{875} & -\frac{549}{2800} & \frac{296}{175} & -\frac{639}{560} & \frac{18}{7} & -\frac{9983}{14000} & -\frac{36}{175} & \frac{9}{800} \\ \frac{216433}{259200} & -\frac{98441}{648000} & \frac{1601467}{1036800} & -\frac{160867}{207360} & \frac{55223}{32400} & -\frac{127253}{259200} & \frac{3462053}{5184000} & \frac{57281}{2073600} \\ \frac{11776}{14175} & -\frac{1856}{14175} & \frac{20992}{14175} & -\frac{1816}{2835} & \frac{20992}{14175} & -\frac{1856}{14175} & \frac{11776}{14175} & \frac{1978}{14175} \end{pmatrix}$$

https://dx.doi.org/10.4314/njt.v45i1.16

$$D_0 = \begin{pmatrix} -\frac{8183}{57600} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{81}{200} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{225}{256} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{47}{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{225}{256} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{81}{200} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8183}{57600} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Ten-Point Hybrid Block Method

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41198923}{383201280} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1837879}{14968800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3016537}{23654400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{243119}{1871100} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10068395}{76640256} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4891}{36960} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{36409331}{273715200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{62462}{467775} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1056117}{7884800} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{80335}{598752} \end{pmatrix}$$

https://dx.doi.org/10.4314/njt.v45i1.16

$$B_0=\begin{pmatrix}
\frac{4776367597}{53648179200} & \frac{342968359}{638668800} & -\frac{14221079}{31933440} & \frac{13395017}{35481600} & -\frac{170469863}{638668800} & \frac{5199599}{35481600} & -\frac{9471361}{159667200} & \frac{14825963}{894136320} & -\frac{21860567}{7664025600} & \frac{435569}{1916006400} \\
\frac{56921}{49896} & \frac{38245531}{34927200} & -\frac{228779}{103950} & \frac{3518051}{2494800} & -\frac{110857}{124740} & \frac{228551}{498960} & -\frac{55553}{311850} & \frac{8951}{184800} & -\frac{213883}{26195400} & \frac{3197}{4989600} \\
\frac{7875621}{7884800} & -\frac{12115251}{7884800} & -\frac{175537}{157696} & \frac{3492747}{788480} & -\frac{1650699}{788480} & \frac{3782837}{3942400} & -\frac{1379727}{3942400} & \frac{144099}{1576960} & -\frac{14173}{946176} & \frac{9081}{7884800} \\
\frac{444688}{467775} & -\frac{27604}{31185} & \frac{812864}{155925} & \frac{14692}{17325} & -\frac{891872}{155925} & \frac{34096}{17325} & -\frac{19904}{31185} & \frac{97459}{623700} & -\frac{11504}{467775} & \frac{172}{93555} \\
\frac{2254625}{2433024} & -\frac{5701375}{8515584} & \frac{7428625}{2128896} & -\frac{91617875}{12773376} & \frac{89035}{49896} & \frac{63825875}{12773376} & -\frac{7757875}{6386688} & \frac{6752125}{25546752} & -\frac{5977375}{153280512} & \frac{71495}{25546752} \\
\frac{2007}{2200} & -\frac{4941}{8800} & \frac{2243}{770} & -\frac{127629}{30800} & \frac{71523}{7700} & -\frac{93133}{30800} & -\frac{11313}{3850} & \frac{5913}{12320} & -\frac{2909}{46200} & \frac{261}{61600} \\
\frac{9882859}{10948608} & -\frac{9061717}{18247680} & \frac{119736841}{45619200} & -\frac{15892219}{5068800} & \frac{51662317}{9123840} & -\frac{6696389}{1013760} & \frac{6182071}{1824768} & \frac{103219333}{91238400} & -\frac{30425129}{273715200} & \frac{1818929}{273715200} \\
\frac{977792}{1091475} & -\frac{70736}{155925} & \frac{42496}{17325} & -\frac{82136}{31185} & \frac{46336}{10395} & -\frac{559136}{155925} & \frac{697856}{155925} & -\frac{1637458}{1091475} & -\frac{4736}{18711} & \frac{592}{51975} \\
\frac{28086669}{31539200} & -\frac{4656123}{11038720} & \frac{4600881}{1971200} & -\frac{9155511}{3942400} & \frac{30244023}{7884800} & -\frac{10066113}{3942400} & \frac{1072359}{394240} & -\frac{7429239}{7884800} & \frac{176321121}{220774400} & \frac{42039}{1576960} \\
\frac{132875}{149688} & -\frac{80875}{199584} & \frac{28375}{12474} & -\frac{24125}{11088} & \frac{89035}{24948} & -\frac{24125}{11088} & \frac{28375}{12474} & -\frac{80875}{199584} & \frac{132875}{149688} & \frac{80335}{598752}
\end{pmatrix}$$

$$D_0=\begin{pmatrix}
-\frac{4671}{28672} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{592}{945} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{8967}{5120} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{202025}{55296} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{8967}{5120} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{592}{945} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4671}{28672} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Twelve-Point Hybrid Block Method

$$B_1=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{217828596101}{2092278988800} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9685910551}{81729648000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5296955957}{43051008000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{45727447}{364864500} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{10603337425}{83691159552} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1716787}{13453440} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{27383624947}{213497856000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{82209104}{638512875} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1853086281}{14350336000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7693565}{59439744} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3524030807}{27172454400} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1364651}{10510500}
\end{pmatrix},$$

https://dx.doi.org/10.4314/njt.v45i1.16

$$B_0=\begin{pmatrix}
\frac{2399005227481}{199717539840000} & \frac{143694128647}{193729536000} & -\frac{805565958997}{1046139494400} & \frac{585715738313}{697426329600} & -\frac{92058146539}{116237721600} & \frac{379155993019}{622702080000} & -\frac{322356534833}{871782912000} & \frac{280366029637}{1627328102400} & -\frac{248058519763}{4184557977600} & \frac{44485966699}{3138418483200} & -\frac{555378503}{264176640000} & \frac{1523489833}{10461394944000} \\
\frac{351532411}{283783500} & \frac{6193982891137}{3432645216000} & -\frac{1956849793}{510810300} & \frac{817500157}{259459200} & -\frac{1126349711}{425675250} & \frac{3714390517}{1945944000} & -\frac{157970737}{141891750} & \frac{2748634547}{5448643200} & -\frac{608086051}{3575672100} & \frac{26929829}{672672000} & -\frac{6419629}{1094593500} & \frac{32874301}{81729648000} \\
\frac{701883423}{652288000} & -\frac{2405295531}{1025024000} & -\frac{2134341942661}{602714112000} & \frac{5693148801}{574013440} & -\frac{22420612377}{3587584000} & \frac{293549813}{73216000} & -\frac{15736609071}{7175168000} & \frac{13694593833}{14350336000} & -\frac{38480839}{123002880} & \frac{3624134949}{50226176000} & -\frac{298686483}{28700672000} & \frac{30351653}{43051008000} \\
\frac{130397552}{127702575} & -\frac{102138818}{70945875} & \frac{1115013808}{127702575} & \frac{31074134183}{5959453500} & -\frac{1213164448}{70945875} & \frac{50055988}{6081075} & -\frac{121716256}{30405375} & \frac{92690287}{56756700} & -\frac{328190608}{638512875} & \frac{73596206}{638512875} & -\frac{323504}{19864845} & \frac{2773777}{2554051500} \\
\frac{3080234825}{3099672576} & -\frac{47645141275}{41845579776} & \frac{239420823625}{41845579776} & -\frac{155362431625}{9299017728} & -\frac{13040377615}{6974263296} & \frac{20815681175}{996323328} & -\frac{17713299025}{2324754432} & \frac{77081331125}{27897053184} & -\frac{68232505375}{83691159552} & \frac{816074845}{4649508864} & -\frac{91719925}{3804143616} & \frac{131821825}{83691159552} \\
\frac{8559801}{8758750} & -\frac{110750391}{112112000} & \frac{33723}{7150} & -\frac{222472143}{22422400} & \frac{938601}{35035} & -\frac{522602}{125125} & -\frac{16172001}{875875} & \frac{112624047}{22422400} & -\frac{1384769}{1051050} & \frac{170361}{640640} & -\frac{307161}{8758750} & \frac{249719}{112112000} \\
\frac{9377671793}{9704448000} & -\frac{10644650681}{11860992000} & \frac{179948062903}{42699571200} & -\frac{109056090941}{14233190400} & \frac{94567606301}{5930496000} & -\frac{520325850541}{17791488000} & \frac{181408126403}{17791488000} & \frac{56023586353}{4744396800} & -\frac{49585701361}{21349785600} & \frac{44478283871}{106748928000} & -\frac{407325877}{7907328000} & \frac{670574947}{213497856000} \\
\frac{476012288}{496621125} & -\frac{534558944}{638512875} & \frac{2498491648}{638512875} & -\frac{18537164}{2837835} & \frac{2624218624}{212837625} & -\frac{504264512}{30405375} & \frac{1803899392}{70945875} & -\frac{15067424612}{1489863375} & -\frac{27238144}{5108103} & \frac{50920736}{70945875} & -\frac{50370304}{638512875} & \frac{2879516}{638512875} \\
\frac{5468187447}{5740134400} & -\frac{39884007717}{50226176000} & \frac{5326612839}{1435033600} & -\frac{11999625039}{2050048000} & \frac{75481061679}{7175168000} & -\frac{253440171}{20500480} & \frac{52292838681}{3587584000} & -\frac{42526300293}{2870067200} & \frac{1394321074647}{200904704000} & \frac{11657086893}{7175168000} & -\frac{2493909}{18636800} & \frac{97551513}{14350336000} \\
\frac{58114775}{61297236} & -\frac{585775}{768768} & \frac{510470875}{143026884} & -\frac{1177668875}{217945728} & \frac{10715525}{1135134} & -\frac{22822175}{2223936} & \frac{37366075}{3405402} & -\frac{584807875}{72648576} & \frac{21103375}{2918916} & -\frac{69376948565}{27461161728} & -\frac{673175}{2270268} & \frac{673175}{59439744} \\
\frac{249467004743}{264176640000} & -\frac{1050095462053}{1426553856000} & \frac{1315576674137}{380414361600} & -\frac{750245585903}{147938918400} & \frac{137841955031}{15850598400} & -\frac{507341820689}{56609280000} & \frac{9832266037}{1078272000} & -\frac{363857356027}{63402393600} & \frac{396489743903}{95103590400} & -\frac{5155520689}{3522355200} & \frac{185778184209959}{199717539840000} & \frac{24466579093}{951035904000} \\
\frac{75024}{79625} & -\frac{632322}{875875} & \frac{1786256}{525525} & -\frac{3432753}{700700} & \frac{7293024}{875875} & -\frac{1045204}{125125} & \frac{7293024}{875875} & -\frac{3432753}{700700} & \frac{1786256}{525525} & -\frac{632322}{875875} & \frac{75024}{79625} & \frac{1364651}{10510500}
\end{pmatrix}$$

$$D_0=\begin{pmatrix}
-\frac{2224234463}{12192768000} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{673175}{762048} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{30469149}{10035200} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{670472}{99225} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3607625}{331776} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{353721}{28000} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3607625}{331776} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{670472}{99225} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{30469149}{10035200} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{673175}{762048} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2224234463}{12192768000} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

2.1 Order, Error Constants and Stability of Hybrid Block Method (4)

In this section, the fundamental properties of Equation (4) are examined to determine the order, error constants, and stability of the hybrid block method. The order p of the method is established based on the following condition:

Let $e=(1,1,\cdots,1)^T$ and $c=(v_1,v_2,\cdots,v_k)^T$.

$$C_0=e-A_1e$$
$$C_1=c^1-B_1e-B_0e$$
$$C_2=c^2-2B_0c^1-2D_0c^0$$
$$C_3=c^3-3B_0c^2-6D_0c^1$$
$$C_4=c^4-4B_0c^3-12D_0c^2$$

Therefore: $C_j=c^j-j\,B_0c^{j-1}-j(j-1)D_0c^{j-2},\quad j=5,6,\cdots$

Note that the powers of the vectors of C_j are component-wise. The block method (4) is of order p, and $C_j=0$ for $j=0,1,\cdots,p$, $C_{p+1}\neq 0$.

Lemma 1

The order of hybrid block method (4) is p = 2k + 2.

Proof

Suppose that Z_m is the exact solution to (1), then the local truncation error E_m is given as

$$\|E_m\|=Z_m-Y_m-A_1Y_{m-1}-h\,B_1F(Y_{m-1})-hB_0F(Y_m)-h^2D_0F'(Y_m)$$
$$=C_{p+1}h^{2k+3}Y^{(2k+3)}(t)+\mathrm{O}(h^{2k+4}) \qquad (10)$$

where $\|\cdot\|$ is the maximum norm and $C_{p+1}h^{2k+3}Y^{(2k+3)}(t)$ is the principal truncation error. For the two-point block method, the local truncation error for each constituent scheme is $\left(-\frac{1}{5760}h^5y^{(5)}(t)+O(h^6),\ -\frac{1}{2880}h^5y^{(5)}(t)+O(h^6)\right)^T$, the error constant is $C_5=\left(-\frac{1}{5760},-\frac{1}{2880}\right)^T$ and the order is $p=(4,4)^T$.

□

The error constants are shown in Table 1 for the hybrid block methods.

https://dx.doi.org/10.4314/njt.v45i1.16

Table 1: Error constants of hybrid block method (4)

k	Error Constant
1	$\left(-\frac{1}{5760},-\frac{1}{2880}\right)^T$
2	$\left(-\frac{107}{7741440},-\frac{1}{30240},-\frac{3}{57344},-\frac{1}{15120}\right)^T$
3	$\left(-\frac{6031}{3715891200},-\frac{233}{58060800},-\frac{9}{1433600},-\frac{31}{3628800},-\frac{1625}{148635648},-\frac{9}{716800}\right)^T$
4	$\left(-\frac{1129981}{4904976384000},,-\frac{22063}{38320128000},-\frac{3649}{4037017600},-\frac{37}{29937600},-\frac{61525}{39239811072},-\frac{299}{157696000},-\frac{1570597}{700710912000},-\frac{37}{14968800}\right)^T$
5	$\left(-\frac{35661419}{973860765696000},-\frac{15378107}{167382319104000},-\frac{386691}{2671771648000},-\frac{259073}{1307674368000},-\frac{673175}{2678117105664},-\frac{69939}{229605376000},-\frac{156529079}{437243609088000},-\frac{67157}{163459296000},-\frac{6849279}{14694744064000},-\frac{673175}{1339058552832}\right)^T$
6	$\left(-\frac{45183033541}{7198778780024832000},-\frac{110830703}{7030057402368000},-\frac{3745741}{109844646912000},-\frac{356102275}{8227175748599808},-\frac{753}{14350336000},-\frac{1294135129}{20987693236224000},-\frac{486371}{6865290432000},-\frac{263485143}{3291622670336000},-\frac{5015525}{56240459218944},-\frac{64572255121}{654434434547712000},-\frac{753}{7175168000}\right)^T$

***Lemma* 2**

(i) The new hybrid block method (4) is zero-stable.

(ii) The new hybrid block method (4) is convergent.

(iii) The new hybrid block method (4) is A-stable of order $p \leq 14$.

Proof

(i) The zero-stability of (4) is examined as

$$\Rightarrow \det\left(A_0 w - A_1\right) = 0 \quad w^{k-1}(w-1) = 0 \tag{11}$$

By computing the roots of the first characteristic polynomial, it is observed in (11) that all the roots lie inside, with only the principal root on the boundary, of the unit circle. Hence, the hybrid block method is zero-stable.

(ii) The hybrid block method (4) is consistent of order $k \geq 4$, and zero-stable. Hence it is convergent.

(iii) By applying the hybrid block method (4) to the test equations $y' = -\lambda y$ and $y'' = -\lambda^2 y$, it yields the stability polynomial

$$\pi(w,z) = \det(A_0 Y_m - A_1 Y_{m-1} - hB_1 F_{m-1} - h B_0 F_m - h^2 D_0 F_m)\,. \tag{12}$$

The boundary locus technique was deployed to determine the regions of absolute stability of the method (4). It was established that the new hybrid block method (4) attains A-stability for $k \leq 6$ as presented in Fig. 1. The figure shows that the *k*-point hybrid block method (4) is A-stable of order $p \leq 14$ as the stability region $\Re_\Delta$ encompasses the left half of the complex plane, $\Re_\Delta \supseteq \{z \mid \mathrm{Re}(z) < 0\}$.

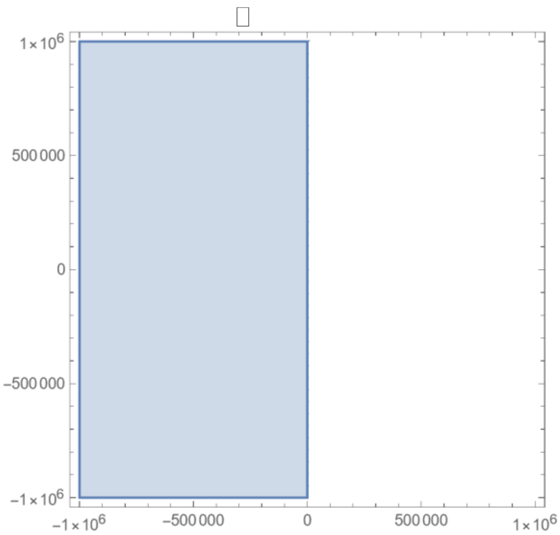

Figure 1. Boundary locus plot of hybrid block method (4) for $p \leq 14$

3.0 IMPLEMENTATION, NUMERICAL RESULTS AND THEIR DISCUSSION

In this section, the implementation of the hybrid block method (4) on linear and non-linear IVPs is outlined. The numerical findings of the hybrid block methods are presented to demonstrate their superiority and effectiveness for the solution of stiff IVPs in ODEs. The computations were done using

https://dx.doi.org/10.4314/njt.v45i1.16

our custom code implemented in the MATLAB R2021b environment on a 64-bit computer. The application of (4) to integrate a system of stiff IVPs (1) results into algebraic system of the form

$$A_0 Y_m = hB_0 F(Y_m) + h^2 D_0 F'(Y_m) + \chi_{m-1}, \qquad (13)$$

where $\chi_{m-1} = A_1 Y_{m-1} + hB_1 F(Y_{m-1}) + h^2 D_1 F'(Y_{m-1})$ is a known function of previously computed set of values. This equation must be solved for the vector Y_m at each cycle of integration and this can be solved by fixed point iteration or by solving

$$F(Y_m) = A_0 Y_m - hB_0 F(Y_m) - h^2 D_0 F'(Y_{m-1}) - \chi_{m-1} = 0, \qquad (14)$$

by some form of Newton's iteration. By solving the algebraic system (13) using the fixed point iteration destroys the good stability property of the implicitness of such numerical method. Thus, to maintain their good stability property, its implicitness is better resolved by using any modified Newton-Raphson's technique, [13]. The Newton iteration takes the form

$$Y_m^{[r+1]} - Y_m^{[r]} = -J^{-1} F(Y_m^{[r]}) \qquad (15)$$

where $r = 0, 1, \cdots$ and the matrix J is the Jacobian matrix.

Numerical results generated by the family of methods are compared to numerical solutions generated by the following methods proposed in the literature:

i. The GCE2BD5 in [19].
ii. The multi-derivative multistep method in [20].
iii. Order six block integrators in [21].
iv. Second derivative methods of class 1 and class 2 in [22].

Test Problems

The following test problems are outlined for the numerical experiment of the methods developed in this study. Existing exact solutions are used as benchmarks for verifying the accuracy of the method, [23]. Comparisons on the basis of absolute error was made using the method, [24].

Test Problem 1 **(cf. [18], [20])**

Consider an epidemiological model that provides insight into the dynamics of infectious diseases in a closed population over time given as

$$S' = \mu(1-S) - \beta IS \qquad (16)$$
$$I' = -I(\mu+\gamma) + \beta IS \qquad (17)$$
$$R' = -\mu R + \gamma I \qquad (18)$$

where the parameters μ, β and γ are positive. Define $y(t) = S + I + R$, such that S, I and R denote susceptible, infected and recovered individuals respectively. Adding (16), (17) and (18) results to

$$y' = \mu(1-y), \quad y(0) = \tfrac{1}{2}, \qquad \mu = \tfrac{1}{2}. \qquad (19)$$

The exact solution of (19) is given $y(t) = 1 - \mu e^{-\mu t}$.

The absolute error over the finite interval 0 ≤ t ≤ 1 for the IVP (19) was recorded when the hybrid block method (4) of order 6 was used for its integration. This was compared to the methods in [21] and [20] of the same order using *h* = 0.1. The numerical results are shown in Table 2 and the solution plot is shown in Fig 2.

Test Problem 2 **(cf. [22], [19])**

Consider the initial value problem

$$y_1' = -\alpha y_1 - \beta y_2 + (\alpha+\beta-1)e^{-t}, \qquad y_1(0) = 1$$
$$y_2' = \beta y_1 - \alpha y_2 + (\alpha-\beta-1)e^{-t}, \qquad y_2(0) = 1$$

with exact solution $y_1(t) = y_2(t) = e^{-t}$ and $y_3(t) = t$.

The hybrid block method (4) of order 8 was applied to integrate this problem at various points using step size *h* = 0.09. The numerical results of this experiment are shown in Table 3 along with results from methods proposed in the literature. The solution plot is shown in Fig 3.

Table 2. Absolute Error $\| y_i(t) - y_{ih} \|$ of Test Problem 1

t	Method in [21] *p* = 6	Method in [20] *p* = 6	New Hybrid *p* = 6

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0.1	2.0×10^{-11}	3.766×10^{-12}	2.331×10^{-14}
0.2	3.0×10^{-11}	2.498×10^{-12}	2.331×10^{-14}
0.3	1.0×10^{-10}	3.013×10^{-12}	4.219×10^{-14}
0.4	2.0×10^{-10}	2.408×10^{-12}	4.219×10^{-14}
0.5	1.0×10^{-10}	5.374×10^{-12}	5.729×10^{-14}
0.6	2.0×10^{-10}	4.225×10^{-12}	5.729×10^{-14}
0.7	1.0×10^{-10}	4.538×10^{-12}	6.017×10^{-14}
0.8	2.0×10^{-10}	3.943×10^{-12}	6.917×10^{-14}
0.9	3.0×10^{-10}	6.274×10^{-12}	7.827×10^{-14}
1.0	3.0×10^{-10}	5.242×10^{-12}	7.827×10^{-14}

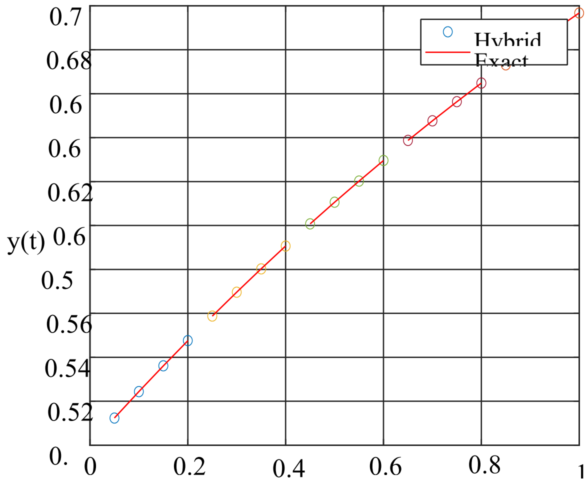

Figure 2. Solution Plot for Test Problem 1

Table 3. Absolute Error $\| y_i(t) - y_{ih} \|$ of Test Problem 2

t	*yi*	Class 1 in [22] *p* = 8	Class 2 in [22] *p* = 8	GCE2BD in [19] p = 8	New Hybrid p = 8
4.5	y_1	0.1×10^{-10}	0.1×10^{-10}	0.6×10^{-14}	0.1×10^{-16}
	y_2	0.1×10^{-10}	0.1×10^{-10}	0.8×10^{-15}	0.6×10^{-17}
9.0	y_1	0.1×10^{-12}	0.1×10^{-12}	0.3×10^{-16}	0.8×10^{-19}
	y_2	0.1×10^{-12}	0.1×10^{-12}	0.1×10^{-16}	0.1×10^{-18}
13.5	y_1	0.1×10^{-15}	0.1×10^{-11}	0.8×10^{-18}	0.1×10^{-20}
	y_2	0.1×10^{-15}	0.1×10^{-11}	0.5×10^{-18}	0.6×10^{-21}
18.0	y_1	0.1×10^{-17}	0.1×10^{-11}	0.1×10^{-19}	0.8×10^{-23}
	y_2	0.1×10^{-17}	0.1×10^{-11}	0.2×10^{-20}	0.1×10^{-23}

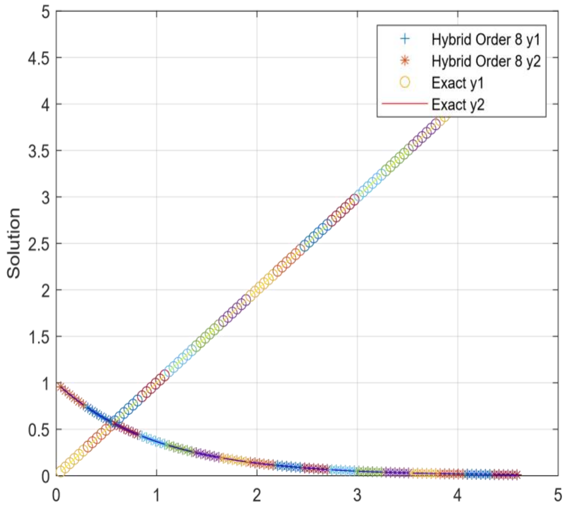

Figure 3. Solution Plot for Test Problem 2, $h = 0.09$, $t = 4.5$

3.1 Discussion of Results

For Test Problem 1, the hybrid block method of order 6 was applied and the absolute error recorded over the finite interval $0 \le t \le 1$. The result of the hybrid block method was compared to the method in [21] and the method in [20] of the same order using h = 0.1. The hybrid block method outperformed the methods in comparison. The results are shown in Table 2, and the solution plot in Fig 2. For Test Problem 2, the hybrid block method of order 8 was applied to integrate the problem at various points using step size h = 0.09. The results were compared to the GCE2BD5 in [19] and the E2BD in [22]. The new hybrid block method showed superior accuracy than the methods in comparison. The numerical results are shown in Table 3 and solution plot in Fig. 3.

4.0 CONCLUSION

https://dx.doi.org/10.4314/njt.v45i1.16

In this study, a family of hybrid block methods was developed for solving stiff initial value problems (IVPs) in ordinary differential equations (ODEs). These methods were constructed using interpolation and collocation techniques to enhance their stability and accuracy. It was demonstrated that the proposed family includes methods that are A-stable for orders up to $p \leq 14$, making them highly effective for stiff

systems. Numerical experiments confirmed that the new methods outperformed existing approaches in terms of accuracy, particularly for challenging stiff problems. Additionally, the hybrid block structure provided computational efficiency by allowing simultaneous evaluation at multiple points. These results highlight the potential of the proposed methods for improving the numerical treatment of stiff ODEs.

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